Limite de șiruri: UTILE

1) $\lim_{n\to\infty} n = \lim_{n\to\infty} n^2 = +\infty,\ \lim_{n\to\infty}\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^2} = 0,\ \lim_{n\to\infty}\frac{a}{b\cdot n+c} = 0,\ b\neq 0, \lim_{n\to\infty} n^{\alpha} = \begin{cases} +\infty, & daca\,\alpha>0 \\ 1, & daca\,\alpha=0 \\ 0, & daca\,\alpha<0 \end{cases}$

Exemple.

a) $\lim_{n\to\infty}\frac{7}{3\cdot n+2}$; b) $\lim_{n\to\infty}\frac{11}{n^2+7}$; c) $\lim_{n\to\infty}\frac{5}{9n}$; d) $\lim_{n\to\infty}\left(2n^2+3n\right)$; e) $\lim_{n\to\infty}\left(-2n+2015\right)$.

2) $\lim_{n\to\infty}\frac{a\cdot n^p+b\cdot n^{p-1}+...}{c\cdot n^q+d\cdot n^{q-1}+...} = \begin{cases} \frac{a}{c}, & daca\ p=q \\ 0, & daca\,q>p \\ \frac{a}{c}\cdot\infty, & daca\ p>q \end{cases}$

Exemple.

a) $\lim_{n\to\infty}\frac{2n+11}{3n+7}$; b) a) $\lim_{n\to\infty}\frac{-3n^4+3n^2+121}{3n+9}$; c) $\lim_{n\to\infty}\frac{2n^4+3n^2+121}{4n^4-11}$; d) $\lim_{n\to\infty}\frac{-7n^4+12}{3n^6+9}$; e) $\lim_{n\to\infty}\frac{n^4+3n^2}{-2n+9}$.

3) $\lim_{n\to\infty} q^n = \begin{cases} +\infty, q>1 \\ 1, q=1 \\ 0, -1<q<1 \\ nu\, exista, q\leq -1 \end{cases}, \ q\in\mathbb{R}$

Exemple.

a) $\lim_{n\to\infty} 3^n$; b) $\lim_{n\to\infty}\left(\frac{3}{2}\right)^n$; c) $\lim_{n\to\infty}\left(\frac{1}{5}\right)^n$; d) $\lim_{n\to\infty}\left(\sqrt{2}\right)^n$; e) $\lim_{n\to\infty}\left(\sqrt{3}-\sqrt{2}\right)^n$.

4) $a_n\leq b_n\leq c_n$ și $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = x \Rightarrow \lim_{n\to\infty} b_n = x$ (**Criteriul cleștelui**)

Exemple.

a) $\lim_{n\to\infty}\frac{\sin n}{n}$; b) $\lim_{n\to\infty}\sqrt[n]{3^n+4^n}$; c) $\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+...+\frac{n}{n^2+n}\right)$; d) $\lim_{n\to\infty}\frac{[x]+[2x]+...+[nx]}{n^2}$;
e) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+...+\frac{1}{\sqrt{n^2+n}}\right)$; f) $\lim_{n\to\infty}\left(\frac{1}{n^2+1}+\frac{2}{n^2+2}+...+\frac{n}{n^2+n}\right)$; g) $\lim_{n\to\infty}\frac{[an]}{n}$.

5) $a_n\leq b_n$ și $\lim_{n\to\infty} a_n = +\infty \Rightarrow \lim_{n\to\infty} b_n = +\infty$.

Exemple.

a) $\lim_{n\to\infty} n^4$; b) $\lim_{n\to\infty}\sqrt{n^3}$; c) $\lim_{n\to\infty} 3^n$; d) $\lim_{n\to\infty} 3^n$; e) $\lim_{n\to\infty}\sqrt[5]{n^7}$; f) $\lim_{n\to\infty}\frac{n^2}{n+1}$.

6) $0 \le a_n \le b_n$ și $\lim\limits_{n\to\infty} b_n = 0 \Rightarrow \lim\limits_{n\to\infty} a_n = 0$.

Exemple.

a) $\lim\limits_{n\to\infty}\frac{1}{n^7}$; b) $\lim\limits_{n\to\infty}\frac{1}{n!}$; c) $\lim\limits_{n\to\infty}\frac{3^n}{n!}$; d) $\lim\limits_{n\to\infty}\frac{n^2}{n!}$; e) $\lim\limits_{n\to\infty}\frac{n!}{n^n}$.

7) $a_n \to 0$ si b_n marginit $\Rightarrow a_n \cdot b_n \to 0$.

Exemple.

a) $\lim\limits_{n\to\infty}\frac{1}{n}\cdot\cos\left(\frac{n\pi}{2}\right)$; b) $\lim\limits_{n\to\infty}\frac{1}{2^n}\cdot\sin n$; c) $\lim\limits_{n\to\infty}\frac{1}{n^2}\cdot\sin n$; d) $\lim\limits_{n\to\infty}\frac{1}{3^n}\cdot\frac{n+1}{n}$; e) $\lim\limits_{n\to\infty}\frac{1}{10^n}\cdot\left(\sqrt{n+1}-\sqrt{n}\right)$.

8) $a_n \to +\infty$ si b_n marginit $\Rightarrow a_n + b_n \to +\infty$

Exemple.

a) $\lim\limits_{n\to\infty} 7^n + \cos\left(\frac{n\pi}{3}\right)$; b) $\lim\limits_{n\to\infty}\left(\frac{11}{2}\right)^n + \frac{n}{3n+1}$; c) $\lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^n + 2^n$; d) $\lim\limits_{n\to\infty}\left(n^4 + \frac{n+1}{n}\right)$; e) $\lim\limits_{n\to\infty}\left(n + \sqrt[n]{2}\right)$.

9)
$$a_n \to 0,\ b_n \to 0 \Rightarrow a_n + b_n \to 0$$
$$a_n \cdot b_n \to 0$$
$$\alpha \cdot a_n \to 0,\ \alpha \in \mathbb{R}$$

Dacă șirul a_n de numere reale pozitive este strict crescător și nemărginit, atunci $\frac{1}{a_n} \to 0$.

Exemple.

a) $\lim\limits_{n\to\infty}\left(\frac{1}{n} + (0,3)^n\right)$; b) $\lim\limits_{n\to\infty}\left(\frac{1}{n^2} + \left(\frac{1}{2}\right)^n\right)$; c) $\lim\limits_{n\to\infty}\frac{1}{n^5}\cdot\left(\frac{3}{4}\right)^n$; d) $\lim\limits_{n\to\infty}\frac{3}{n^3\cdot 7^n}$; e) $\lim\limits_{n\to\infty}\frac{2}{n^3}\left(\frac{-11}{2n+5} + \frac{2}{n^2+8}\right)$.

10) **Operații în** $\mathbb{R}$ **:**

$a - \infty = -\infty, a + \infty = +\infty, +\infty + \infty = +\infty, -\infty - \infty = -\infty; 6\cdot\infty = \infty, -6\cdot\infty = -\infty; \frac{a}{\pm\infty} = 0, a \in \mathbb{R}$

Cazuri de nedeterminare:

$\infty - \infty; 0\cdot(\pm\infty); \frac{0}{0}; \frac{\pm\infty}{\pm\infty}; 0^0; 1^{\pm\infty}; (\pm\infty)^0$

11) **Studiați mărginirea următoarelor șiruri:**

a) $x_n = \frac{n}{n+1}, n \ge 1$; b) $x_n = \frac{2\cdot 3^n + 4^n}{3\cdot 4^n + 5^n}, n \ge 0$; c) $x_n = \sqrt{n+1} - \sqrt{n}, n \ge 1$; d) $x_n = \frac{n!}{n^n}, n \ge 1$; e) $x_0 = \sqrt{2}, x_n = \sqrt{2 + x_{n-1}}, n \ge 1$; f) $x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}, n \ge 1$; g) $x_0 = 0, x_n - x_{n-1} = \frac{n}{(n+1)!}, n \ge 1$.

12) **Studiați monotonia următoarelor șiruri:**

a) $x_n = \frac{1}{n+1}, n \ge 1$; b) $x_n = \frac{n+1}{n}, n \ge 1$; c) $x_n = \frac{n^2}{n+1}, n \ge 1$; d) $x_n = \frac{n^2}{n^2-1}, n \ge 2$; e) $x_n = \sqrt{n+1} - \sqrt{n}, n \ge 0$; f) $x_n = \sqrt{n^2+1} - n, n \ge 1$; g) $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}, n \ge 1$; h) $x_n = n\left(\frac{1}{2^n}\right), n \ge 1$;

i) $x_n = \frac{3^n}{n!}, n \ge 3$; j) $x_n = \sum\limits_{k=1}^{n}\log_{\frac{1}{2}}\frac{k^2+2k}{(k+1)^2}, n \ge 1$; k) $x_n = \frac{5^n}{n!}, n \ge 5$; l) $x_0 = \frac{1}{4}, x_{n+1} = \frac{1}{4} + x_n^{\ 2}, n \ge 0$.